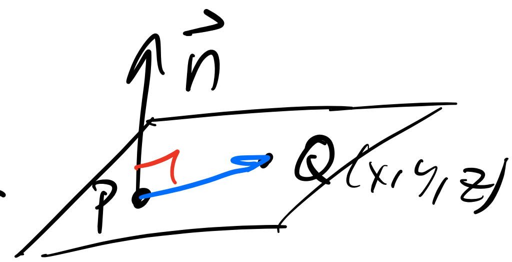
Due Wed

3.3 – Orthogonality

Definition 1: Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to every vector in R^n.

A vector $\mathbf{n}$ that is orthogonal to a line in R^2 or R^3 or a plane in R^3 is called a **normal**.

3. Find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal.
$P(-1, 2, -1)$, $\mathbf{n} = (-2, 1, -1)$



Let Q (x, y, z) be any other point in the plane.

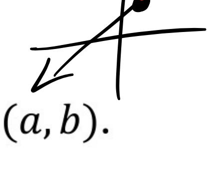
Then $\vec{n} \cdot \overrightarrow{PQ} = 0$

$\Rightarrow (-2, 1, -1) \cdot (x+1, y-2, z+1) = 0$

$-2(x+1) + (y-2) - (z+1) = 0$

$\Rightarrow -2x + y - z = 5$

Theorem 3.3.1



a) If a and b are constants that are not both zero, then an equation of the form
$ax + by + c = 0$ represents a line in R^2 with normal $\mathbf{n} = (a, b)$.

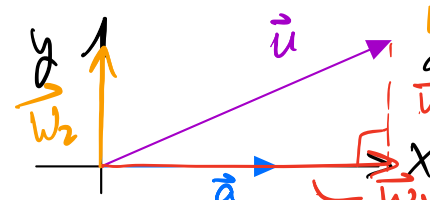
b) If a, b, and c are constants that are not all zero, then an equation of the form
$ax + by + cz + d = 0$ represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

Theorem 3.4.3 (not a typo)
If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.

Consider the homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad & \vdots \quad\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

R^2



$\vec{w}_2 \perp \vec{a}$ and $\vec{w}_1 + \vec{w}_2 = \vec{u}$

$\vec{w}_1 = \mathrm{proj}_{\vec{a}}\, \vec{u}$

Shifting gears,

Theorem 3.3.2 Projection Theorem
If $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^n, and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

$$\vec{w}_1 = k\vec{a} \Rightarrow \vec{u}\cdot\vec{a} = (\vec{w}_1 + \vec{w}_2)\cdot\vec{a}$$

$$= (k\vec{a} + \vec{w}_2)\cdot\vec{a}$$

$$= k\,\vec{a}\cdot\vec{a} = k\|\vec{a}\|^2$$

$$k = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2} \Rightarrow \boxed{\vec{w}_1 = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}}$$

When $\mathbf{u}$, a vector in R^n, is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of a vector $\mathbf{a}$ in R^n, $\mathbf{w}_1 = \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **orthogonal projection of u on a** or the **vector component of u along a**. The vector $\mathbf{w}_2 = \mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **vector component of u orthogonal to a**.

$\vec{u} - \vec{w}_1 = \vec{w}_2$

19. Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a.**

$\mathbf{u} = (2, 1, 1, 2)$, $\mathbf{a} = (4, -4, 2, -2)$

$$\vec{w}_1 = \text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a} = \frac{8-4+2-4}{16+16+4+4} \vec{a}$$

$$= \frac{2}{40} \vec{a} = \frac{1}{20} \vec{a}$$

$$\boxed{\vec{w}_1 = \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10}\right)}$$

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = (2, 1, 1, 2) - \left(\frac{1}{5}, -\frac{1}{5}, \frac{1}{10}, -\frac{1}{10}\right)$$

$$\boxed{\vec{w}_2 = \left(\frac{9}{5}, \frac{6}{5}, \frac{9}{10}, \frac{21}{10}\right)}$$

Checks: $\vec{w}_2 \cdot \vec{a} = \left(\frac{9}{5}, \frac{6}{5}, \frac{9}{10}, \frac{21}{10}\right) \cdot (4, -4, 2, -2)$

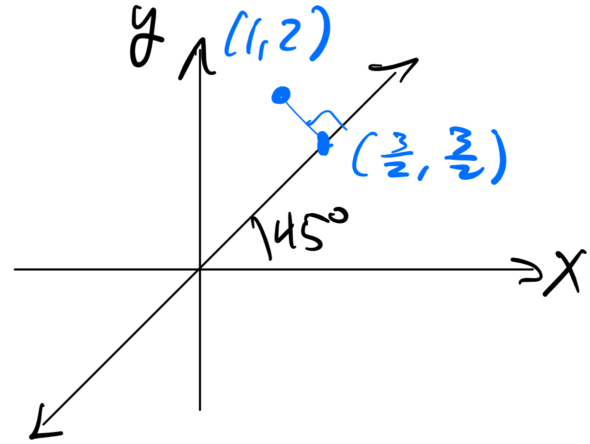
$\vec{w}_1 + \vec{w}_2 = (2, 1, 1, 2) = \vec{u}$ ✓ $\quad = \frac{36}{5} - \frac{24}{5} + \frac{9}{5} - \frac{21}{5} = 0$ ✓

38. Find the standard matrix for the orthogonal projection of R^2 onto the stated line, and then use that matrix to find the orthogonal projection of the given point onto that line.

The orthogonal projection of $(1, 2)$ onto the line that makes an angle of $\pi/4$ $(= 45°)$ with the positive x-axis.

$$A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \right]$$

Find the image of T_A on $\vec{e}_1$ & $\vec{e}_2$



$\vec{a} = (\cos\theta, \sin\theta)$

$$\text{proj}_{\vec{a}}\, \vec{e}_1 = \frac{\vec{a}\cdot\vec{e}_1}{\|\vec{a}\|^2}\,\vec{a}$$

$$= \cos\theta\,(\cos\theta, \sin\theta) = (\cos^2\theta, \sin\theta\cos\theta)$$

Likewise, $\text{proj}_{\vec{a}}\, \vec{e}_2 = (\sin\theta\cos\theta\ ,\ \sin^2\theta)$

$$A = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

for $\theta = 45°$, $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

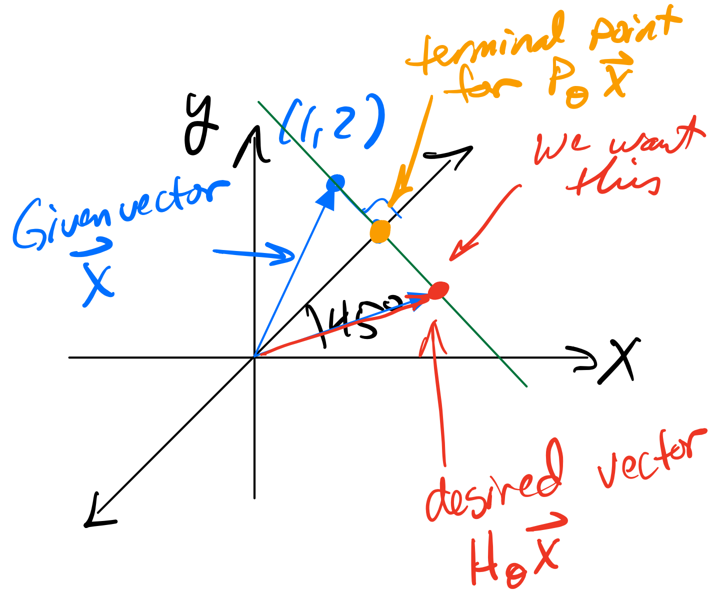
In general, the matrix A is denoted

$$P_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$

36. Find the standard matrix for the reflection of R^2 about the stated line, and then use that matrix to find the reflection of the given point about that line.

The reflection of $(1, 2)$ about the line that makes an angle of $\pi/4$ $(= 45°)$ with the positive x-axis.

The distance between $P_\theta \vec{x}$ and $\vec{x}$ is half the distance between $H_\theta \vec{x}$ and $\vec{x}$.



Thus $P_\theta \vec{x} - \vec{x} = \frac{1}{2}(H_\theta \vec{x} - \vec{x})$

$$\Rightarrow H_\theta \vec{x} = 2P_\theta \vec{x} - \vec{x} = (2P_\theta - I)\vec{x}$$

$$\Rightarrow H_\theta = 2P_\theta - I = \begin{bmatrix} 2\cos^2\theta - 1 & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$H_{\pi/4} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So $(2,1)$ is the desired point.

Theorem 3.3.3 Theorem of Pythagoras in R^n

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in R^n with the Euclidean inner product, then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

But $\vec{u}\cdot\vec{v} = 0$ so $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ ✓